Is True Of – Part 1: Russell Redux

This month's post and the next one are based on the YouTube video entitled [Russell's Paradox - a simple explanation of a profound problem](https://youtu.be/ymGt7I4Yn3k) by Jeffery Kaplan. In that video, Kaplan links the well-known mathematical paradox found by Bertrand Russell with familiar linguistic paradoxes in a what a friend characterized as a "stimulating alternative perspective".

[Russell’s paradox](http://aristotle2digital.blogwyrm.com/?p=594) shook the foundations of mathematics by casting doubt on whether it were possible to logically establish mathematics as an objective discipline with rules that reflect something beyond human convention or construction. And while, in the aftermath, a variety of patches were proposed that sidestep the issue by eliminating certain constructions, Kaplan's essential point is that the same mental processes that lead to Russell's paradox lead to logical paradoxes in natural language. These natural language paradoxes, in turn, reflect something deeper in how we think and, as a result, we can't sidestep these processes in everyday life the way they are currently and narrowly sidestepped in mathematics. Paradoxes seem to be built into the fabric of human existence.

To be clear, his essential point is not novel and the connections that exist in human thought between formal logic, mathematical logic, and linguistics have been covered in within a variety of contexts including the [liar's paradox](http://aristotle2digital.blogwyrm.com/?p=209). What is intriguing about Kaplan's analysis is the method he employs to make this point using the basic function of predication within natural language. I don't know if his argument is truly one of his own making or it originates elsewhere but it is a clever and particularly clear way of seeing that logic can only carry one so far in the world.

Following Kaplan, we'll start with a review of Russell's paradox. The paradox arises in naive set theory as a function of three basic ideas. First is the notion of a set as a collection of any objects of our perception or of our thought. Second is the idea that set composition is unlimited in scope, concrete or intangible, real or imagined, localized or spread over time and space. Kaplan calls this idea Rule Number 1 - Unrestricted Composition. Third is the idea that what matters for a set is what elements contains not how those elements are labeled. Set membership can be determined by a specific listing or by specifying some comprehension rule that describes the members globally. Kaplan calls this idea Rule Number 2 - Set Identity is Determined by Membership.

Using Rules 1 and 2, Kaplan then outlines a set of rules 3-11, each springing in some way from Rules 1 or 2 as a parent, which is indicated inside the parentheses:

* Rule 3 - Order Doesn't Matter (Rule 2)
* Rule 4 - Repeats Don't Change Anything (Rule 2)
* Rule 5 - Description Doesn't Matter (Rule 2)
* Rule 6 - The Union of Any Sets is a Set (Rule 1)
* Rule 7 - Any Subset is a Set (Rule 1)
* Rule 8 - A Set Can Have Just One Member (Rule 1)
* Rule 9 - A Set Can Have No Members (Rules 1 & 2)
* Rule 10 - You Can Have Sets of Set (Rule 1)
* Rule 11 - Sets Can Contain Themselves (Rule 1)

Kaplan walks the viewer, in an amusing way, through increasing more complex set construction examples using these 11 rules, although I'll modify the elements used in the examples to be more to my liking. His construction starts with an example of a finite, listed set:

\[ A = \{ \mathrm{Frodo}, \mathrm{Sam}, \mathrm{Merry}, \mathrm{Pippin} \} \; .\]

Employing Rule 2 allows us to rewrite set $A$ in an equivalent way as

\[ A = \{ x | x \mathrm{\;is\;a\;hobbit\;in\;the\;Fellowship\;of\;the\;Ring} \} \; .\]

Kaplan then gives the famous example of a set used thought by Gottlob Frege and Bertrand Russell as a candidate for the fundamental definition of what the ordinal number 1 is:

\[ \{ x | x \mathrm{\;of\;singleton\;sets} \} \; .\]

Here we have a set, that if listed explicitly, might start as

\[ \{ \{\mathrm{Frodo}\}, \{\mathrm{Sam}\}, \{\mathrm{Merry}\}, \{\mathrm{Pippin}\}, \{\mathrm{Empire\;State\;Building}\} \ldots \} \; .\]

All of this seems plausible if not particularly motivated, but the wheels fall off when we look at Rule 11. That rule is deceptively simple in that it is easy to say the words ‘sets can contain themselves’ but is ultimately difficult, perhaps impossible, to understand what those words mean as there exists no constructive way to actually build a set that contains itself. Membership is done by a comprehension rule expressed as a sentence; any sentence will do and quoting Kaplan: "If you can think of it, you can throw it in a set." We'll return to that sentiment next month when we talk about Kaplan's language-based analog to Russell's paradox. We'll call such a Rule-11 set an extraordinary set and any set not containing itself ordinary. The next step is then to define the set-of-all-extraordinary sets as

\[ \{ x | x \mathrm{\;is\;a\;set\;that\;contains\;itself} \} \; .\]

This set, while having no constructive way of being listed, is still considered a valid set by naive set theory. As preposterous as this set's existence may be it generates no paradoxes. The real heartbreaker is the set defined as

\[ \{ x | x \mathrm{\;is\;a\;set\;that\;does\;not\;contains\;itself} \} \; .\]

This set of all ordinary sets has no well-defined truth value. If we assume it does contain itself then it must obey the comprehension rule 'does not contain itself' so then it doesn't contain itself. Alternatively, if we assume it does not contain itself then, in order to be inside itself, it must be true that it meets the inclusion criterion that it doesn’t contain itself. And, thus, the paradox is born.

Mathematicians have patched set theory by 'changing the rules' (as Kaplan puts it). They developed different systems wherein Rule 11 or some equivalent is expressly forbidden (e.g., Zermelo- Fraenkel with [the Axiom of Choice](http://aristotle2digital.blogwyrm.com/?p=1009)).

But, Kaplan objects, the rules of set theory are not made up but are objective rules that reflect the object, common rules of thought and language that we all use. He gives a linguistic argument based on the act of predication in natural language to make this point. That analysis is the subject of next month’s posting.